

Asymptotics for d -dimensional Lévy-type processes

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Abstract

We consider a general d -dimensional Lévy-type process with killing. Combining the classical Dyson series approach with a novel polynomial expansion of the generator $\mathcal{A}(t)$ of the Lévy-type process, we derive a family of asymptotic approximations for transition densities and European-style options prices. Examples of stochastic volatility models with jumps are provided in order to illustrate the numerical accuracy of our approach. The methods described in this paper extend the results from Corielli et al. (2010), Pagliarani and Pascucci (2013) and Lorig et al. (2013a) for Markov diffusions to Markov processes with jumps.

Keywords: multi-dimensional Lévy-type process with killing, asymptotic approximation, integro-differential equation

1 Introduction

In a multi-dimensional Markovian setting, the time evolution of a market model is usually described by the solution X of a Lévy-Itô stochastic differential equation (SDE). Such a model allows for features commonly seen in markets, such as stochastic volatility, jumps, default, co-integration and correlation. Many quantities of interest (e.g., option prices, net present values) can be expressed as expectations of the form $u(t, x) := \mathbb{E}[\varphi(X_T) | X_t = x]$. Under mild conditions, the function $u(t, x)$ is the unique classical solution of a partial integro-differential equation (PIDE). Unfortunately, closed form and even semi-closed form solutions of these PIDEs are available only in rare cases. As such, it is important to develop general methods for finding analytical approximations for the solutions of these PIDEs.

Within the mathematical finance literature, a number of different approaches have been taken for finding approximate transition densities and option prices for markets described by Markov processes. Most of these techniques involve expansions that exploit a small parameter or a limiting case. For example, Benhamou et al. (2009) develop analytical approximations for models with local volatility and Gaussian jumps in the small diffusion and small jump frequency/size limits (see also the recent review paper

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by Bompis and Gobet (2013)). Deuschel et al. (2014) obtain densities for diffusion processes in a small noise limit. Fouque et al. (2011) find option prices for Black-Scholes-like multiscale models where volatility is driven by two factors, one running on a fast scale, one running on a slow scale. Lorig (2012); Lorig and Lozano-Carbassé (2013) extend these multiscale techniques to more general diffusions and to the exponential Lévy setting.

Recently, Pagliarani and Pascucci (2012) introduce a method for finding asymptotic solutions of parabolic PDEs. The approach, called the *adjoint expansion method*, is extended by Pagliarani et al. (2013); Lorig et al. (2014a) to models with jumps and it was further generalized by Lorig et al. (2013a) to a family of asymptotic expansions for a d -dimensional market described by an Itô SDE (i.e., a Markov market with no jumps). The method consists of expanding the pricing PDE in polynomial basis functions, which results in a nested sequence of Cauchy problems, and deriving analytical solutions for these nested Cauchy problems. In this paper, we extend the results of Pagliarani et al. (2013); Lorig et al. (2014a, 2013a) to the PIDEs that arise when markets are described by a d -dimensional Lévy-Itô SDE. Results presented here also simplify results from Pagliarani et al. (2013); Lorig et al. (2014a, 2013a).

The rest of this paper proceeds as follows. In Section 2 we present a general d -dimensional market model. We also describe the kinds of derivative-assets we wish to price, and we relate the price of such derivative-assets to the solution of a parabolic PIDE. In Section 3 we introduce the idea of polynomial expansions of the pricing PIDE and in Section 4, we derive a family of analytical price approximations – one for each polynomial expansion of the pricing PIDE. Lastly, in Section 5 we provide a numerical example, illustrating the versatility and accuracy of our methods.

2 Market model

We take, as given, an equivalent martingale measure \mathbb{Q} defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$. All stochastic processes defined below live on this probability space and all expectations are taken with respect to \mathbb{Q} . The risk-neutral dynamics of our market are described by the following d -dimensional Markov Lévy-type process

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t + \int_{\mathbb{R}^d} z d\tilde{N}(t, X_{t-}, dt, dz).$$

Here W is a standard m -dimensional Brownian motion, and $\tilde{N}(\cdot, \cdot, dt, dz)$, given by

$$\tilde{N}(t, x, dt, dz) = N(t, x, dt, dz) - \nu(t, x, dz)dt, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

is a family of compensated Poisson measures on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. The drift vector μ and volatility matrix σ map $\mu : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, respectively. We assume the Lévy kernel ν satisfies

$$\int_{\mathbb{R}^d} \min\{|z|, |z|^2\} \bar{\nu}(dz) < \infty, \quad \bar{\nu}(dz) := \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \nu(t, x, dz), \quad (2.1)$$

which is rather standard for Lévy-type models. The components of X could represent a number of things such as e.g., economic factors, asset prices, indices, or functions of these quantities. We assume a risk-free

interest rate of the form $r(t, X_t)$ where $r : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$. We also introduce a random time ζ , which is given by

$$\zeta = \inf \left\{ t \geq 0 : \int_0^t \gamma(s, X_s) ds \geq \mathcal{E} \right\}, \quad \gamma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+,$$

with \mathcal{E} exponentially distributed and independent of X . The random time ζ could represent the default time of an asset, the arrival of an economic shock, etc..

Denote by V the no-arbitrage price of a European derivative expiring at time T with payoff

$$H(X_T) \mathbb{I}_{\{\zeta > T\}} + G(X_T) \mathbb{I}_{\{\zeta \leq T\}} = (H(X_T) - G(X_T)) \mathbb{I}_{\{\zeta > T\}} + G(X_T).$$

It is well known (see, for instance, Jeanblanc et al. (2009)) that

$$V_t = \mathbb{E} \left[e^{-\int_t^T r(s, X_s) ds} G(X_T) | X_t \right] + \mathbb{I}_{\{\zeta > t\}} \mathbb{E} \left[e^{-\int_t^T (r(s, X_s) + \gamma(s, X_s)) ds} (H(X_T) - G(X_T)) | X_t \right], \quad t < T. \quad (2.2)$$

Thus, to value a European-style option, one must compute functions of the form

$$u(t, x) := \mathbb{E} \left[e^{-\int_t^T \lambda(s, X_s) ds} \varphi(X_T) \mid X_t = x \right]. \quad (2.3)$$

Under mild assumptions (see, for instance, Pascucci (2011)), the function u , defined by (2.3), satisfies the Kolmogorov backward equation

$$(\partial_t + \mathcal{A}(t))u = 0, \quad u(T, x) = \varphi(x), \quad x \in \mathbb{R}^d, \quad (2.4)$$

where the operator $\mathcal{A}(t)$ is given explicitly by

$$\mathcal{A}(t) = \int_{\mathbb{R}^d} \nu(t, x, dz) \left(e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle \right) + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^T)_{ij}(t, x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^d \mu_i(t, x) \partial_{x_i} - \lambda(t, x), \quad (2.5)$$

with

$$\langle z, x \rangle := \sum_{i=1}^d z_i x_i, \quad \nabla_x := (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_d}), \quad e^{\langle z, \nabla_x \rangle} f(x) := f(x + z).$$

The formal representation of the *shift operator* $e^{\langle z, \nabla_x \rangle}$ is motivated by the fact that its Taylor expansion applied to the function $f(x)$ gives the Taylor expansion of $f(x+z)$ about the point x . As in (Øksendal and Sulem, 2005, Chapter 1), we regard the domain of $\mathcal{A}(t)$ to be all functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\mathcal{A}(t)f(x)$ exists and is finite for all $x \in \mathbb{R}^d$.

Remark 2.1 (Martingale property). Let us denote by $X^{(i)}$ the i th component of the vector X and assume that

$$\int_{|z| \geq 1} e^{z_i} \bar{\nu}(dz) < \infty,$$

for some $i \leq d$, with $\bar{\nu}$ as in (2.1). If $S_t := \mathbb{I}_{\{\zeta > t\}} e^{X^{(i)}}$ is supposed to be a traded asset then, in order for S to be a martingale, the drift μ_i must satisfy

$$\mu_i(t, x) = \gamma(t, x) - \int_{\mathbb{R}^d} \nu(t, x, dz) (e^{z_i} - 1 - z_i) - \frac{1}{2} (\sigma \sigma^T)_{ii}(t, x),$$

To see this, set $H(x) = e^{x_i}$, $G(x) = 0$ and impose $V_t = S_t$ in (2.2).

3 General expansion basis

Let us start by rewriting the differential operator (2.5) in the more compact form

$$\mathcal{A}(t) := \int_{\mathbb{R}^d} \nu(t, x, dz) \left(e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle \right) + \sum_{|\alpha| \leq 2} a_\alpha(t, x) D_x^\alpha, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d,$$

where by standard notations

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d, \quad |\alpha| = \sum_{i=1}^d \alpha_i, \quad D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}.$$

In this section we introduce a family of expansion schemes for $\mathcal{A}(t)$, which we shall use to construct closed-form approximate solutions (one for each family) of (2.4).

Definition 3.1. For $|\alpha| \leq 2$ and $n \leq N \in \mathbb{N}_0$, let $a_{\alpha,n} = a_{\alpha,n}(t, x)$ and $\nu_n = \nu_n(t, x, dz)$ be such that the following hold:

- (i) For any $t \in [0, T]$, $a_{\alpha,n}(t, \cdot)$ are polynomial functions with $a_{\alpha,0}(t, x) \equiv a_{\alpha,0}(t)$, and for any $x \in \mathbb{R}^d$ the functions $a_{\alpha,n}(\cdot, x)$ belong to $L^\infty([0, T])$.
- ii) For any $t \in [0, T]$, $x \in \mathbb{R}^d$, we have

$$\nu_n(t, x, dz) = \sum_{|\beta| \leq M_n} x^\beta \nu_{n,\beta}(t, dz), \quad M_n \in \mathbb{N}_0, \quad (3.1)$$

where each $\nu_{n,\beta}(t, dz)$ satisfies condition (2.1). Moreover, $M_0 = 0$, $\nu_0 \geq 0$ and

$$\int_{|z| \geq 1} e^{\lambda|z|} \nu_0(t, dz) < \infty, \quad t \in [0, T], \quad (3.2)$$

for some positive λ .

Then we say that $(\mathcal{A}_n(t))_{0 \leq n \leq N}$, defined by

$$\begin{aligned} \mathcal{A}_n(t)f(x) &= \sum_{|\beta| \leq M_n} x^\beta \int_{\mathbb{R}^d} \nu_{n,\beta}(t, dz) \left(e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle \right) f(x) + \sum_{|\alpha| \leq 2} a_{\alpha,n}(t, x) D_x^\alpha f(x) \\ &\equiv \int_{\mathbb{R}^d} \nu_n(t, x, dz) \left(e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle \right) f(x) + \sum_{|\alpha| \leq 2} a_{\alpha,n}(t, x) D_x^\alpha f(x), \end{aligned} \quad (3.3)$$

is an N th order polynomial expansion of $\mathcal{A}(t)$.

Definition 3.1 allows for very general polynomial specifications. The idea is to choose an expansion $(\mathcal{A}_n(t))$ that closely approximates $\mathcal{A}(t)$. The precise sense of this approximation will depend on the application. Below, we present three polynomial expansions. The first two expansion schemes provide an accurate approximation $\mathcal{A}(t)$ in a pointwise local sense, under the assumption of smooth coefficients. The last expansion scheme approximates $\mathcal{A}(t)$ in a global sense and can be applied even in the case of discontinuous coefficients.

Example 3.2. (Taylor polynomial expansion)

Assume the coefficients $a_\alpha(t, \cdot) \in C^N(\mathbb{R}^d)$ and that the compensator ν takes the form

$$\nu(t, x, dz) = h(t, x, z) \bar{\nu}(dz)$$

where $h(t, \cdot, z) \in C^N(\mathbb{R}^d)$ with $h \geq 0$, and $\bar{\nu}$ is a Lévy measure. Then, for any fixed $\bar{x} \in \mathbb{R}^d$ and $n \leq N$, we define ν_n and $a_{\alpha,n}$ as the n th order term of the Taylor expansions of ν and a_α respectively in the spatial variables x around the point \bar{x} . That is, we set

$$\begin{aligned} \nu_n(t, x, dz) &= \sum_{|\beta|=n} \frac{D_x^\beta h(t, \bar{x}, z)}{\beta!} (x - \bar{x})^\beta \bar{\nu}(dz), \\ a_{\alpha,n}(t, x) &= \sum_{|\beta|=n} \frac{D_x^\beta a_\alpha(t, \bar{x})}{\beta!} (x - \bar{x})^\beta, \end{aligned} \quad |\alpha| \leq 2,$$

where as usual $\beta! = \beta_1! \cdots \beta_d!$ and $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$. The expansion proposed in Lorig et al. (2013b) and Lorig et al. (2014c) is the particular case when $\nu \equiv 0$, whereas the expansion proposed in Lorig et al. (2014a) and Lorig et al. (2014b) is a particular case when $d = 1$.

Example 3.3. (Time-dependent Taylor polynomial expansion)

Under the assumptions of Example 3.2, fix a trajectory $\bar{x} : \mathbb{R}_+ \rightarrow \mathbb{R}^d$. We then define $\nu_n(t, x, dz)$ and $a_{\alpha,n}(t, x)$ as the n th order term of the Taylor expansions of $\nu(t, x, dz)$ and $a_\alpha(t, x)$ respectively around $\bar{x}(t)$. More precisely, we set

$$\begin{aligned} \nu_n(t, x, dz) &= \sum_{|\beta|=n} \frac{D_x^\beta h(t, \bar{x}(t), z)}{\beta!} (x - \bar{x}(t))^\beta \bar{\nu}(dz), \\ a_{\alpha,n}(t, x) &= \sum_{|\beta|=n} \frac{D_x^\beta a_\alpha(t, \bar{x}(t))}{\beta!} (x - \bar{x}(t))^\beta, \end{aligned} \quad |\alpha| \leq 2.$$

This expansion for the coefficients allows the expansion point \bar{x} of the Taylor series to evolve in time according to the evolution of the underlying process X_t . For instance, one could choose $\bar{x}(t) = \mathbb{E}[X_t]$. In Lorig et al. (2013b) this choice results in a highly accurate approximation for option prices and implied volatility in the Heston (1993) model.

Example 3.4. (Hermite polynomial expansion)

Hermite expansions can be useful when the diffusion coefficients are discontinuous. A remarkable example in financial mathematics is given by the Dupire's local volatility formula for models with jumps (see Friz et al. (2013)). In some cases, e.g., the well-known Variance-Gamma model, the fundamental solution (i.e., the transition density of the underlying stochastic model) has singularities. In such cases, it is natural to approximate it in some L^p norm rather than in the pointwise sense. For the Hermite expansion centered at \bar{x} , one sets

$$\begin{aligned} \nu_n(t, x, dz) &= \sum_{|\beta|=n} \langle \mathbf{H}_\beta(\cdot - \bar{x}), \nu(t, \cdot, dz) \rangle_\Gamma \mathbf{H}_\beta(x - \bar{x}), \\ a_{\alpha,n}(t, x) &= \sum_{|\beta|=n} \langle \mathbf{H}_\beta(\cdot - \bar{x}), a_\alpha(t, \cdot) \rangle_\Gamma \mathbf{H}_\beta(x - \bar{x}), \end{aligned} \quad |\alpha| \leq 2,$$

where the inner product $\langle \cdot, \cdot \rangle_\Gamma$ is an integral over \mathbb{R}^d with a Gaussian weighting centered at \bar{x} and the functions $\mathbf{H}_\beta(x) = H_{\beta_1}(x_1) \cdots H_{\beta_d}(x_d)$ where H_n is the n -th one-dimensional Hermite polynomial (properly normalized so that $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle_\Gamma = \delta_{\alpha,\beta}$ with $\delta_{\alpha,\beta}$ being the Kronecker's delta function).

4 Formal solution via Dyson series

In this section we present a heuristic argument to pass from an expansion of the operator $\mathcal{A}(t)$ in (2.5) to an expansion for u , the solution of problem (2.4). The following argument is not intended to be rigorous. Rather, the computations that follow provide motivation for the price expansion given in Definition 4.1. Throughout this section, we will generally omit x -dependence, except where it is needed for clarity. To begin, we presume that the operator $\mathcal{A}(t)$ can be formally written as

$$\mathcal{A}(t) = \mathcal{A}_0(t) + \mathcal{B}(t), \quad \mathcal{B}(t) = \sum_{n=1}^{\infty} \mathcal{A}_n(t). \quad (4.1)$$

We insert expansion (4.1) for $\mathcal{A}(t)$ into Cauchy problem (2.4) and find

$$(\partial_t + \mathcal{A}_0(t))u(t) = -\mathcal{B}(t)u(t), \quad u(T) = \varphi.$$

Note that, by construction, $\mathcal{A}_0(t)$ is the generator of an additive process. Therefore, by Duhamel's principle, we have

$$u(t) = \mathcal{P}_0(t, T)\varphi + \int_t^T dt_1 \mathcal{P}_0(t, t_1)\mathcal{B}(t_1)u(t_1), \quad (4.2)$$

where $\mathcal{P}_0(t, T)$ is the semigroup of operators generated by $\mathcal{A}_0(t)$. Inserting expression (4.2) for u into the right-hand side of (4.2) and iterating we obtain

$$\begin{aligned} u(t) &= \mathcal{P}_0(t, T)\varphi + \int_t^T dt_1 \mathcal{P}_0(t, t_1)\mathcal{B}(t_1)\mathcal{P}_0(t_1, T)\varphi \\ &\quad + \int_t^T dt_1 \int_{t_1}^T dt_2 \mathcal{P}_0(t, t_1)\mathcal{B}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{B}(t_2)u(t_2) \\ &= \dots \\ &= \mathcal{P}_0(t, T)\varphi + \sum_{k=1}^{\infty} \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \\ &\quad \mathcal{P}_0(t, t_1)\mathcal{B}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{B}(t_2) \cdots \mathcal{P}_0(t_{k-1}, t_k)\mathcal{B}(t_k)\mathcal{P}_0(t_k, T)\varphi \end{aligned} \quad (4.3)$$

$$\begin{aligned} &= \mathcal{P}_0(t, T)\varphi + \sum_{n=1}^{\infty} \sum_{k=1}^n \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \\ &\quad \sum_{i \in I_{n,k}} \mathcal{P}_0(t, t_1)\mathcal{A}_{i_1}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{A}_{i_2}(t_2) \cdots \mathcal{P}_0(t_{k-1}, t_k)\mathcal{A}_{i_k}(t_k)\mathcal{P}_0(t_k, T)\varphi, \end{aligned} \quad (4.4)$$

$$I_{n,k} = \{i = (i_1, i_2, \dots, i_k) \in \mathbb{N}^k \mid i_1 + i_2 + \cdots + i_k = n\}. \quad (4.5)$$

The second-to-last equality (4.3) is known as the *Dyson series expansion* of u (see, for instance, Section 5.7 of Sakurai and Tuan (1994) or Chapter IX.2.6 of Kato (1995)). To obtain (4.4) from (4.3) we have used

(4.1) to replace $\mathcal{B}(t)$ by the infinite sum $\sum_{n=1}^{\infty} \mathcal{A}_n(t)$, and we have partitioned on the sum of the subscripts of the (\mathcal{A}_{i_k}) . Expansion (4.4) motivates the following definition.

Definition 4.1. For a fixed N th order polynomial expansion $(\mathcal{A}_n(t))_{0 \leq n \leq N}$ satisfying Definition 3.1, we define \bar{u}_N , the N th order price approximation of u , as

$$\bar{u}_N := \sum_{n=0}^N u_n, \quad (4.6)$$

where

$$\begin{aligned} u_0(t) &:= \mathcal{P}_0(t, T)\varphi, \\ u_n(t) &:= \sum_{k=1}^n \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \\ &\quad \sum_{i \in I_{n,k}} \mathcal{P}_0(t, t_1) \mathcal{A}_{i_1}(t_1) \mathcal{P}_0(t_1, t_2) \mathcal{A}_{i_2}(t_2) \cdots \mathcal{P}_0(t_{k-1}, t_k) \mathcal{A}_{i_k}(t_k) \mathcal{P}_0(t_k, T) \varphi, \quad n \geq 1. \end{aligned} \quad (4.7)$$

Here, $\mathcal{P}_0(t, T)$ is the semigroup generated by $\mathcal{A}_0(t)$ and $I_{n,k}$ is as given in (4.5).

In Sections 4.1 and 4.2 we will provide explicit expressions for u_0 and $(u_n)_{n \geq 1}$ respectively.

4.1 Expression for u_0

In what follows, it will be helpful to recall the definition of the Fourier and inverse Fourier transforms. For any function φ in the Schwartz class, we define

$$\begin{aligned} \text{Fourier transform:} \quad & \mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^d} dx \varphi(x) e^{i\langle \xi, x \rangle}, \\ \text{Inverse transform:} \quad & \mathcal{F}^{-1}[\hat{\varphi}](x) = \varphi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \hat{\varphi}(\xi) e^{-i\langle \xi, x \rangle}. \end{aligned}$$

Recall that by construction $M_0 = 0$ (cf. Definition 3.1) and therefore the operator $\mathcal{A}_0(t)$ has time-dependent coefficients which are independent of x . Then the action of the semigroup of operators $\mathcal{P}_0(t, T)$ of $\mathcal{A}_0(t)$ is well-known:

$$u_0(t) := \mathcal{P}_0(t, T)\varphi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{P}_0(t, x, T, \xi) \hat{\varphi}(-\xi) d\xi \quad (4.8)$$

where

$$\hat{P}_0(t, x, T, \xi) := e^{i\langle \xi, x \rangle + \Phi_0(t, T, \xi)} \quad (4.9)$$

with

$$\Phi_0(t, T, \xi) = \sum_{|\alpha| \leq 2} (i\xi)^\alpha \int_t^T ds a_{\alpha,0}(s) + \Psi_0(t, T, \xi), \quad (4.10)$$

and

$$\Psi_0(t, T, \xi) = \int_t^T \int_{\mathbb{R}^d} \left(e^{i\langle \xi, z \rangle} - 1 - i\langle \xi, z \rangle \right) \nu_0(s, dz) ds.$$

Remark 4.2. We introduce \hat{P} and e_ξ , the *characteristic function* and *oscillating exponential*, respectively

$$\hat{P}(t, x, T, \xi) := \mathbb{E} \left[e^{\int_t^T a_{0,0}(s, X_s) ds} e^{i\langle \xi, X_T \rangle} | X_t = x \right], \quad e_\xi(x) = e^{i\langle \xi, x \rangle}, \quad (4.11)$$

where $a_{0,0}$ is short-hand for $a_{(0,0,\dots,0),0}$. From (2.3) we observe that $\hat{P}(t, x, T, \xi)$ is obtained as the special case $\varphi = e_\xi$. We note that $\hat{P}_0(t, x, T, \xi)$ in (4.9) represents the 0th order approximation of $\hat{P}(t, x, T, \xi)$. More generally, we denote by $\hat{P}_n(t, x, T, \xi)$ the n th order approximation of $\hat{P}(t, x, T, \xi)$, obtained by setting $\varphi = e_\xi$ in (4.7).

4.2 Expression for u_n

Remarkably, as the following proposition shows, every $u_n(t)$ can be expressed as a pseudo-differential operator $\mathcal{L}_n(t, T)$ acting on $u_0(t)$.

Proposition 4.3. *Assume that φ belongs to the Schwartz class, and that Φ_0 in (4.10) is a smooth function of the variable ξ . Then the function u_n defined in (4.7) is given explicitly by*

$$u_n(t) = \mathcal{L}_n(t, T)u_0(t), \quad (4.12)$$

where u_0 is given by (4.8) and

$$\mathcal{L}_n(t, T) = \sum_{k=1}^n \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \sum_{i \in I_{n,k}} \mathcal{G}_{i_1}(t, t_1) \mathcal{G}_{i_2}(t, t_2) \cdots \mathcal{G}_{i_k}(t, t_k), \quad (4.13)$$

with $I_{n,k}$ as defined in (4.5) and

$$\begin{aligned} \mathcal{G}_j(t, t_k) &:= \mathcal{A}_j(t_k, \mathcal{M}(t, t_k)) \\ &= \int_{\mathbb{R}^d} \nu_j(t_k, \mathcal{M}(t, t_k), dz) \left(e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle \right) + \sum_{|\alpha| \leq 2} a_{\alpha,j}(t_k, \mathcal{M}(t, t_k)) D_x^\alpha, \end{aligned} \quad (4.14)$$

$$\mathcal{M}(t, t_k) := x + \int_{\mathbb{R}^d} \int_t^{t_k} z \left(e^{\langle z, \nabla_x \rangle} - 1 \right) \nu_0(s, dz) ds + \int_t^{t_k} m(s) ds + \int_t^{t_k} C(s) \nabla_x ds, \quad (4.15)$$

$$\begin{aligned} m(s) &= \begin{pmatrix} a_{(1,0,\dots,0),0}(s) & a_{(0,1,\dots,0),0}(s) & \cdots & a_{(0,0,\dots,1),0}(s) \end{pmatrix}, \\ C(s) &= \begin{pmatrix} 2a_{(2,0,\dots,0),0}(s) & a_{(1,1,\dots,0),0}(s) & \cdots & a_{(0,0,\dots,1),0}(s) \\ a_{(1,1,\dots,0),0}(s) & 2a_{(0,2,\dots,0),0}(s) & \cdots & a_{(0,1,\dots,1),0}(s) \\ \vdots & \vdots & \ddots & \vdots \\ a_{(1,0,\dots,1),0}(s) & a_{(0,1,\dots,1),0}(s) & \cdots & 2a_{(0,0,\dots,2),0}(s) \end{pmatrix}. \end{aligned}$$

Moreover, the components of $\mathcal{M}(t, t_k)$ commute. Therefore the operators $(\mathcal{G}_j(t, t_k))$, which are polynomials in $\mathcal{M}(t, t_k)$ by construction, are well defined.

Proof. The proof consists in showing that the operator $\mathcal{G}_j(t, t_k)$ in (4.14) satisfies

$$\mathcal{P}_0(t, t_k) \mathcal{A}_j(t_k) = \mathcal{G}_j(t, t_k) \mathcal{P}_0(t, t_k). \quad (4.16)$$

Assuming (4.16) holds, we can use the fact that $\mathcal{P}_0(t_k, t_{k+1})$ is a semigroup

$$\mathcal{P}_0(t, T) = \mathcal{P}_0(t, t_1)\mathcal{P}_0(t_1, t_2) \cdots \mathcal{P}_0(t_{k-1}, t_k)\mathcal{P}_0(t_k, T), \quad t \leq t_1 \leq \dots \leq t_k \leq T,$$

and we can re-write (4.7) as

$$u_n(t) = \sum_{k=1}^n \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \sum_{i \in I_{n,k}} \mathcal{G}_{i_1}(t, t_1) \mathcal{G}_{i_2}(t, t_2) \cdots \mathcal{G}_{i_k}(t, t_k) \mathcal{P}_0(t, T) \varphi,$$

from which (4.12)-(4.13) follows directly. Thus, we only need to show that $\mathcal{G}_j(t, t_k)$ satisfies (4.16). It is sufficient to investigate how the operator $\mathcal{P}_0(t, t_k)\mathcal{A}_j(t_k)$ acts on the oscillating exponential in (4.11). First, we note that

$$\mathcal{P}_0(t, t_k)e_\xi(x) = e^{\Phi_0(t, t_k, \xi)}e_\xi(x), \quad (4.17)$$

where $\Phi_0(t, t_k, \xi)$, as given in (4.10), is a smooth function by condition (3.2). Next, we observe that the operator $\mathcal{M}(t, t_k)$ in (4.15) can be written

$$\mathcal{M}(t, t_k) = M(t, t_k, -i\nabla_x), \quad M(t, t_k, \xi) = -i\nabla_\xi(\Phi_0(t, t_k, \xi) + i\langle \xi, x \rangle). \quad (4.18)$$

Denote by \mathcal{M}_j and M_j the j th component of \mathcal{M} and M respectively. Then, using (4.18) we have

$$\begin{aligned} (-i\partial_{\xi_i})(-i\partial_{\xi_j})e^{\Phi_0(t, t_k, \xi)}e_\xi(x) &= (-i\partial_{\xi_i})M_j(t, t_k, \xi)e^{\Phi_0(t, t_k, \xi)}e_\xi(x) \\ &= \mathcal{M}_j(t, t_k)(-i\partial_{\xi_i})e^{\Phi_0(t, t_k, \xi)}e_\xi(x) \\ &= \mathcal{M}_j(t, t_k)M_i(t, t_k, \xi)e^{\Phi_0(t, t_k, \xi)}e_\xi(x) \\ &= \mathcal{M}_j(t, t_k)\mathcal{M}_i(t, t_k)e^{\Phi_0(t, t_k, \xi)}e_\xi(x). \end{aligned} \quad (4.19)$$

More generally for any multi-index β we have

$$(-i\nabla_\xi)^\beta e^{\Phi_0(t, t_k, \xi)}e_\xi(x) = (\mathcal{M}(t, t_k))^\beta e^{\Phi_0(t, t_k, \xi)}e_\xi(x). \quad (4.20)$$

From (4.19) we deduce that operators \mathcal{M}_i and \mathcal{M}_j commute when applied to $e^{\Phi_0(t, t_k, \xi)}e_\xi(x)$, because so do ∂_{ξ_i} and ∂_{ξ_j} . Consequently, \mathcal{M}_i and \mathcal{M}_j also commute when applied to $e_\xi(x)$ or any function that admits a representation as a Fourier transform. To see this observe that

$$\mathcal{M}_j(t, t_k)\mathcal{M}_i(t, t_k)e^{\Phi_0(t, t_k, \xi)}e_\xi(x) = \mathcal{M}_i(t, t_k)\mathcal{M}_j(t, t_k)e^{\Phi_0(t, t_k, \xi)}e_\xi(x).$$

Therefore, since $\mathcal{M}_j(t, t_k)$ acts on x and not ξ we have

$$\mathcal{M}_j(t, t_k)\mathcal{M}_i(t, t_k)e_\xi(x) = \mathcal{M}_i(t, t_k)\mathcal{M}_j(t, t_k)e_\xi(x).$$

Finally, we compute

$$\begin{aligned} \mathcal{P}_0(t, t_k)\mathcal{A}_j(t_k)e_\xi(x) &= \mathcal{P}_0(t, t_k) \int_{\mathbb{R}^d} \nu_j(t_k, x, dz) (e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle) e_\xi(x) \\ &\quad + \sum_{|\alpha| \leq 2} \mathcal{P}_0(t, t_k) a_{\alpha, j}(t_k, x) D_x^\alpha e_\xi(x) \end{aligned} \quad (\text{by (3.3)})$$

$$\begin{aligned}
&= \mathcal{P}_0(t, t_k) \int_{\mathbb{R}^d} (e^{i\langle z, \xi \rangle} - 1 - i\langle z, \xi \rangle) \nu_j(t_k, x, dz) e_\xi(x) \\
&\quad + \sum_{|\alpha| \leq 2} (i\xi)^\alpha \mathcal{P}_0(t, t_k) a_{\alpha, j}(t_k, x) e_\xi(x) \\
&= \int_{\mathbb{R}^d} (e^{i\langle z, \xi \rangle} - 1 - i\langle z, \xi \rangle) \nu_j(t_k, -i\nabla_\xi, dz) \mathcal{P}_0(t, t_k) e_\xi(x) \\
&\quad + \sum_{|\alpha| \leq 2} (i\xi)^\alpha a_{\alpha, j}(t_k, -i\nabla_\xi) \mathcal{P}_0(t, t_k) e_\xi(x) \\
&= \int_{\mathbb{R}^d} (e^{i\langle z, \xi \rangle} - 1 - i\langle z, \xi \rangle) \nu_j(t_k, -i\nabla_\xi, dz) e^{\Phi_0(t, t_k, \xi)} e_\xi(x) \\
&\quad + \sum_{|\alpha| \leq 2} (i\xi)^\alpha a_{\alpha, j}(t_k, -i\nabla_\xi) e^{\Phi_0(t, t_k, \xi)} e_\xi(x) \quad (\text{by (4.17)}) \\
&= \int_{\mathbb{R}^d} (e^{i\langle z, \xi \rangle} - 1 - i\langle z, \xi \rangle) \nu_j(t_k, \mathcal{M}(t, t_k), dz) e^{\Phi_0(t, t_k, \xi)} e_\xi(x) \\
&\quad + \sum_{|\alpha| \leq 2} (i\xi)^\alpha a_{\alpha, j}(t_k, \mathcal{M}(t, t_k)) e^{\Phi_0(t, t_k, \xi)} e_\xi(x) \quad (\text{by (4.20)}) \\
&= \int_{\mathbb{R}^d} \nu_j(t_k, \mathcal{M}(t, t_k), dz) (e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle) e^{\Phi_0(t, t_k, \xi)} e_\xi(x) \\
&\quad + \sum_{|\alpha| \leq 2} a_{\alpha, j}(t_k, \mathcal{M}(t, t_k)) D_x^\alpha e^{\Phi_0(t, t_k, \xi)} e_\xi(x) \\
&= \int_{\mathbb{R}^d} \nu_j(t_k, \mathcal{M}(t, t_k), dz) (e^{\langle z, \nabla_x \rangle} - 1 - \langle z, \nabla_x \rangle) \mathcal{P}_0(t, t_k) e_\xi(x) \\
&\quad + \sum_{|\alpha| \leq 2} a_{\alpha, j}(t_k, \mathcal{M}(t, t_k)) D_x^\alpha \mathcal{P}_0(t, t_k) e_\xi(x) \quad (\text{by (4.17)}) \\
&= \mathcal{G}_j(t, t_k) \mathcal{P}_0(t, t_k) e_\xi(x), \quad (\text{by (4.14)})
\end{aligned}$$

which concludes the proof. \square

Remark 4.4. Error bounds for the Taylor approximation \bar{u}_N in the scalar case $d = 1$ can be found in Lorig et al. (2014a,b).

4.3 Fourier representation for u_n

Using (4.8), (4.9) and (4.12) we have

$$u_n(t, x) = \mathcal{L}_n(t, T) u_0(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\Phi_0(t, T, \xi)} \left(\mathcal{L}_n(t, T) e^{i\langle \xi, x \rangle} \right) \hat{\varphi}(-\xi) d\xi.$$

The term in parenthesis $\mathcal{L}_n(t, T) e^{i\langle \xi, x \rangle}$ can be computed explicitly. However, $\mathcal{L}_n(t, T)$ is, in general, an *integro-differential* operator (when X is a diffusion $\mathcal{L}_n(t, T)$ is simply a differential operator). Thus, for models with jumps, computing $\mathcal{L}_n(t, T) e^{i\langle \xi, x \rangle}$ is a challenge. Remarkably, we will show that there exists a first order *differential* operator $\hat{\mathcal{L}}_n^\xi(t, T)$ such that

$$\mathcal{L}_n^x(t, T) e^{i\langle \xi, x \rangle} = \hat{\mathcal{L}}_n^\xi(t, T) e^{i\langle \xi, x \rangle}, \quad (4.21)$$

where, for clarity, we have explicitly indicated using superscripts that $\mathcal{L}_n^x(t, T)$ acts on x and $\hat{\mathcal{L}}_n^\xi(t, T)$ acts on ξ . With a slight abuse of terminology, we call $\hat{\mathcal{L}}_n^\xi$ the *symbol*¹ of the operator $\mathcal{L}_n^x(t, T)$ in (4.13).

Let us consider the operator $\mathcal{M}^x(t, t_k) \equiv \mathcal{M}(t, t_k)$ in (4.15) and denote by $\mathcal{M}_i^x(t, t_k)$ its i th component. The symbol $\hat{\mathcal{M}}_i^\xi(t, t_k)$ of $\mathcal{M}_i^x(t, t_k)$ is defined analogously to (4.21), that is

$$\mathcal{M}_i^x(t, t_k) e^{i\langle \xi, x \rangle} = \hat{\mathcal{M}}_i^\xi(t, t_k) e^{i\langle \xi, x \rangle}.$$

Explicitly, we have

$$\hat{\mathcal{M}}_i^\xi(t, t_k) = F_i(\xi, t, t_k) - i\partial_{\xi_i}, \quad i = 1, \dots, d,$$

where the function F is defined as

$$F_i(\xi, t, t_k) = \int_{\mathbb{R}^d} \int_t^{t_k} z_i \left(e^{i\langle z, \xi \rangle} - 1 \right) \nu_0(s, dz) ds + \int_t^{t_k} m_i(s) ds + i \int_t^{t_k} (C(s)\xi)_i ds.$$

We note that, while \mathcal{M}^x is a first order *integro-differential* operator, its symbol $\hat{\mathcal{M}}^\xi$ is a first order *differential* operator. For this reason, it is more convenient to use the symbol $\hat{\mathcal{M}}^\xi$ instead of the operator \mathcal{M}^x . Note also that

$$\mathcal{M}_i^x(t, t_k) \mathcal{M}_j^x(t, t_k) e^{i\langle \xi, x \rangle} = \mathcal{M}_i^x(t, t_k) \hat{\mathcal{M}}_j^\xi(t, t_k) e^{i\langle \xi, x \rangle} = \hat{\mathcal{M}}_j^\xi(t, t_k) \mathcal{M}_i^x(t, t_k) e^{i\langle \xi, x \rangle} = \hat{\mathcal{M}}_j^\xi(t, t_k) \hat{\mathcal{M}}_i^\xi(t, t_k) e^{i\langle \xi, x \rangle}.$$

Since \mathcal{M}_i^x and \mathcal{M}_j^x commute when applied to a function that admits a Fourier representation, then $\hat{\mathcal{M}}_j^\xi$ and $\hat{\mathcal{M}}_i^\xi$ also commute when applied to such functions. In particular, the operator $\left(\hat{\mathcal{M}}^\xi(t, t_k) \right)^\beta$, for $\beta \in \mathbb{N}_0^d$, is well defined and we have

$$\left(\hat{\mathcal{M}}^\xi(t, t_k) \right)^\beta e^{i\langle \xi, x \rangle} = (\mathcal{M}(t, t_k))^\beta e^{i\langle \xi, x \rangle}. \quad (4.22)$$

From identity (4.22) we obtain directly the expression of the symbol of \mathcal{G}_j in (4.14). Indeed, recalling the expression (3.1) of ν_j we have

$$\hat{\mathcal{G}}_j^\xi(t, t_k) = \sum_{|\beta| \leq M_j} \int_{\mathbb{R}^d} \left(e^{i\langle z, \xi \rangle} - 1 - i\langle z, \xi \rangle \right) \nu_{j, \beta}(t_k, dz) \left(\hat{\mathcal{M}}^\xi(t, t_k) \right)^\beta + \sum_{|\alpha| \leq 2} (i\xi)^\alpha a_{\alpha, j}(t_k, \hat{\mathcal{M}}^\xi(t, t_k)).$$

Thus we have proved the following lemma

Lemma 4.5. *We have*

$$\hat{\mathcal{L}}_n^\xi(t, T) = \sum_{k=1}^n \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{k-1}}^T dt_k \sum_{i \in I_{n, k}} \hat{\mathcal{G}}_{i_1}^\xi(t, t_1) \hat{\mathcal{G}}_{i_2}^\xi(t, t_2) \cdots \hat{\mathcal{G}}_{i_k}^\xi(t, t_k), \quad (4.23)$$

where $I_{n, k}$ as defined in (4.5).

The following theorem extends the Fourier pricing formula (4.8) to higher order approximations.

¹The operator $\hat{\mathcal{L}}_n^\xi$ is not a function as in the classical theory of pseudo-differential calculus. However $e^{-i\langle \xi, x \rangle} \hat{\mathcal{L}}_n^\xi e^{i\langle \xi, x \rangle}$ is the symbol of $\mathcal{L}_n^x(t, T)$.

Theorem 4.6. Under the assumptions of Proposition 4.3, for any $n \geq 1$ we have

$$u_n(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{P}_n(t, x, T, \xi) \hat{\varphi}(-\xi) d\xi, \quad (4.24)$$

where $\hat{P}_n(t, x, T, \xi)$ is the n th order term of the approximation of the characteristic function of X (cf. Remark 4.2). Explicitly, we have

$$\hat{P}_n(t, x, T, \xi) := \hat{P}_0(t, x, T, \xi) \left(e^{-i\langle \xi, x \rangle} \hat{\mathcal{L}}_n^\xi(t, T) e^{i\langle \xi, x \rangle} \right)$$

where $\hat{P}_0(t, x, T, \xi)$ is the 0th order approximation in (4.9) and $\hat{\mathcal{L}}_n^\xi(t, T)$ is the differential operator defined in (4.23).

Proof. We first note that, since the approximating operator \mathcal{L}_n^x acts in the x variables, then it commutes² with the Fourier pricing operator (4.8). Thus, by (4.12) combined with (4.8), we get

$$\begin{aligned} u_n(t) &= \mathcal{L}_n^x(t, T) u_0(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{L}_n^x(t, T) e^{i\langle \xi, x \rangle + \Phi_0(t, T, \xi)} \hat{\varphi}(-\xi) d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{P}_0(t, x, T, \xi) \left(e^{-i\langle \xi, x \rangle} \hat{\mathcal{L}}_n^\xi(t, T) e^{i\langle \xi, x \rangle} \right) \hat{\varphi}(-\xi) d\xi, \end{aligned}$$

and the thesis follows from (4.21). \square

Remark 4.7. Computing the term in parenthesis above $\left(e^{-i\langle \xi, x \rangle} \hat{\mathcal{L}}_n^\xi(t, T) e^{i\langle \xi, x \rangle} \right)$ is a straightforward exercise since the symbol $\hat{\mathcal{L}}_n^\xi(t, T)$, given in (4.23), is a differential operator.

Remark 4.8. In case of non-integrable payoffs (e.g. Call and Put options), the Fourier representation (4.24) can be easily extended by considering the Fourier transform on the imaginary line $\xi = \xi_r + i\xi_i$. For instance, since the Call option payoff $\varphi(x) = (e^x - e^k)^+$ is not integrable, its Fourier transform $\hat{\varphi}(-\xi)$ must be computed in a generalized sense by fixing an imaginary component of the Fourier variable $\xi_i < -1$.

Remark 4.9. Observe that the N th order approximation (4.6)-(4.24) requires only a single Fourier inversion

$$\bar{u}_N(t, x) = \sum_{n=0}^N u_n(t, x) = \frac{1}{(2\pi)^d} \sum_{n=0}^N \int_{\mathbb{R}^d} \hat{P}_n(t, x, T, \xi) \hat{\varphi}(-\xi) d\xi.$$

Moreover, when evaluating the inverse transform, the number of dimensions over which one must integrate numerically is equal to the number of components of x that appear in the option payoff φ . This is due to the fact that the Fourier transform of a constant is a Dirac delta function. In particular, let $\varphi(x) \equiv \bar{\varphi}(\bar{x})$ with $\bar{x} = (x_1, \dots, x_{d'})$, for some $d' < d$. Then we have $\hat{\varphi}(\xi) = (2\pi)^{d-d'} \hat{\bar{\varphi}}(\bar{\xi}) \delta_0(\xi_{d'+1}) \cdots \delta_0(\xi_d)$ with $\bar{\xi} = (\xi_1, \dots, \xi_{d'})$, and thus

$$\bar{u}_N(t, x) = \frac{1}{(2\pi)^{d'}} \sum_{n=0}^N \int_{\mathbb{R}^{d'}} \hat{P}_n(t, x, T, (\bar{\xi}, 0)) \hat{\bar{\varphi}}(-\bar{\xi}) d\bar{\xi}.$$

²This was one of the main points of the *adjoint expansion method* proposed by Pagliarani et al. (2013).

5 Example: Heston model with stochastic jump-intensity

Consider the following model for an asset $S = e^X$, written under the pricing measure \mathbb{Q} assuming zero interest rates

$$\begin{aligned} dX_t &= \left(-\frac{1}{2} - \int_{\mathbb{R}} \nu(d\zeta)(e^\zeta - 1 - \zeta) \right) Z_t dt + \sqrt{Z_t} dW_t + \int_{\mathbb{R}} \zeta d\tilde{N}(t, Z_t, dt, d\zeta), \\ dZ_t &= \kappa(\theta - Z_t)dt + \delta\sqrt{Z_t}dB_t, \quad d\langle W, B \rangle_t = \rho dt. \end{aligned}$$

Note that, just as in the Heston model, the instantaneous volatility of X is given by $\sqrt{Z_t}$, where Z is a CIR process. Likewise, the instantaneous arrival rate of jumps of size $d\zeta$ is given by $Z_t\nu(d\zeta)$, where ν is a Lévy measure satisfying all of the usual integrability conditions. The generator \mathcal{A} of the process (X, Z) is given by

$$\begin{aligned} \mathcal{A} &= z \left(\mu \partial_x + \frac{1}{2} \partial_x^2 + \int_{\mathbb{R}} \nu(d\zeta)(e^{\zeta \partial_x} - 1 - \zeta \partial_x) \right) + \kappa(\theta - z) \partial_z + \frac{1}{2} \delta^2 z \partial_z^2 + \rho \delta z \partial_x \partial_z, \\ \mu &= -\frac{1}{2} - \int_{\mathbb{R}} \nu(d\zeta)(e^\zeta - 1 - \zeta). \end{aligned}$$

The characteristic function $\hat{P}(t, x, z, T, \xi) := \mathbb{E}[e^{i\xi X_T} | X_t = x, Z_t = z]$ is obtained in Carr and Wu (2004) by expressing the process X as a time-changed Lévy process. One can also obtain the characteristic function by solving for the Fourier transform of the fundamental solution corresponding to the operator $(\partial_t + \mathcal{A})$. We have

$$\begin{aligned} \hat{P}(t, x, z, T, \xi) &= e^{i\xi x + C(T-t, \xi) + z D(T-t, \xi)}, \\ C(\tau, \xi) &= \frac{\kappa\theta}{\delta^2} \left((\kappa - \rho\delta i\xi + d(\xi))\tau - 2 \log \left[\frac{1 - f(\xi)e^{d(\xi)\tau}}{1 - f(\xi)} \right] \right), \\ D(\tau, \xi) &= \frac{\kappa - \rho\delta i\xi + d(\xi)}{\delta^2} \frac{1 - e^{d(\xi)\tau}}{1 - f(\xi)e^{d(\xi)\tau}}, \\ f(\xi) &= \frac{\kappa - \rho\delta i\xi + d(\xi)}{\kappa - \rho\delta i\xi - d(\xi)}, \\ d(\xi) &= \sqrt{-\delta^2 2\psi(\xi) + (\kappa - \rho i\xi \delta)^2}, \\ \psi(\xi) &= i\mu\xi - \frac{1}{2}\xi^2 + \int_{\mathbb{R}} \nu(d\zeta)(e^{i\xi\zeta} - 1 - i\xi\zeta). \end{aligned}$$

With an explicit expression for $\hat{P}(t, x, z, T, \xi)$ available, the price of a European call option can be computed using standard Fourier methods

$$u(t, x, z) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi_r \hat{P}(t, x, z, T, \xi) \hat{\varphi}(-\xi), \quad \hat{\varphi}(\xi) = \frac{-e^{k-i k \xi}}{i\xi + \xi^2}, \quad \xi = \xi_r + i\xi_i, \quad \xi_i < -1. \quad (5.1)$$

Note that, since the call option payoff $\varphi(x) = (e^x - e^k)^+$ is not in $L^1(\mathbb{R})$, its Fourier transform $\hat{\varphi}(\xi)$ must be computed in a generalized sense by fixing an imaginary component of the Fourier variable $\xi_i < -1$.

Also of interest are sensitivities of option prices or *Greeks*. In particular, consider the Δ and the Γ , which are defined as

$$\Delta(t, x, z) := \partial_s u(t, x(s), z) = e^{-x} \partial_x u(t, x, z), \quad (5.2)$$

$$\Gamma(t, x, z) := \partial_s^2 u(t, x(s), z) = e^{-2x} (\partial_x^2 - \partial_x) u(t, x, z), \quad (5.3)$$

where we have used $x(s) = \log s$. When computing terms of the form $\partial_x^m u(t, x, z)$, observe that the differential operator ∂_x^m acts only on the characteristic function \hat{P} appearing in (5.1) and not on the Fourier transform $\hat{\varphi}$ of the payoff φ . Likewise, when using Theorem 4.6 to compute $\partial_x^m \bar{u}_n(t, x, z) = \sum_{i=0}^n \partial_x^m u_i(t, x, z)$ the differential operator ∂_x^m acts only on \hat{P}_i in (4.24).

Now, we specialize to the case where jumps are normally distributed

$$\nu(d\zeta) = \frac{\lambda}{\sqrt{2\pi s^2}} \exp\left(-\frac{(\zeta - m)^2}{2s^2}\right).$$

In Figure 1 we plot the implied volatility σ corresponding to the exact price u as well as the implied volatility $\bar{\sigma}_2$ corresponding to our second order approximation \bar{u}_2 . To compute σ we first compute option prices using (5.1); we then invert the Black-Scholes equation numerically in order to obtain the implied volatility σ . To compute our second order approximation of implied volatility $\bar{\sigma}_2$ we first compute our second order approximation for prices \bar{u}_2 using Theorem 4.6; we then invert the Black-Scholes equation numerically in order to obtain $\bar{\sigma}_2$. Values from Figure 1 can be found in Table 1. In Figure 2 we plot the exact Δ as well as our second order approximation $\bar{\Delta}_2$. In Figure 3 we plot the exact Γ as well as our second order approximation $\bar{\Gamma}_2$. Values from Figures 2 and 3 are given in Tables 2 and 3 respectively. Exact Greeks are computed by combining (5.1), (5.2) and (5.3). Approximate Greeks are computed by combining Theorem 4.6 and equations (5.2) and (5.3).

6 Conclusion

In this paper we derive a family of asymptotic expansions for European option prices when the underlying is modeled as a d -dimensional time inhomogeneous Lévy-type process. By combining the classical Dyson series expansion with a novel polynomial expansion of the generator, we obtain two equivalent representations for approximate option price: (i) as an integro-differential operator acting on the order zero price, and (ii) as a Fourier transform. We implement our pricing approximation on a Heston-like model which allows for both stochastic volatility and stochastic jump intensity. We find that our second order expansion provides an excellent approximation for prices (as seen through corresponding implied volatilities), as well as for the Greeks Δ and Γ .

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	$k - x$	-0.2	-0.15	-0.1	-0.05	0.00	0.05	0.1	0.15	0.2
t=0.10	σ	0.2797	0.2478	0.2269	0.2133	0.2028	0.1940	0.1881	0.1960	0.2296
	$\bar{\sigma}_2$	0.2795	0.2483	0.2271	0.2132	0.2028	0.1939	0.1877	0.1963	0.2324
	rel. err.	0.0006	0.0018	0.0009	0.0003	0.0002	0.0001	0.0020	0.0018	0.0120
t=0.25	σ	0.2441	0.2323	0.2217	0.2120	0.2028	0.1941	0.1863	0.1805	0.1803
	$\bar{\sigma}_2$	0.2456	0.2328	0.2215	0.2116	0.2025	0.1939	0.1859	0.1793	0.1799
	rel. err.	0.0059	0.0018	0.0013	0.0020	0.0013	0.0009	0.0021	0.0067	0.0027
t=0.50	σ	0.2348	0.2266	0.2183	0.2101	0.202	0.1940	0.1864	0.1796	0.1743
	$\bar{\sigma}_2$	0.2350	0.2254	0.2168	0.2088	0.201	0.1933	0.1856	0.1783	0.1723
	rel. err.	0.0005	0.0049	0.0069	0.0063	0.004	0.0037	0.0040	0.0070	0.0116
t=1.00	σ	0.2268	0.2204	0.2138	0.2072	0.2005	0.1939	0.1875	0.1813	0.1757
	$\bar{\sigma}_2$	0.2217	0.2149	0.2089	0.2031	0.1973	0.1914	0.1854	0.1794	0.1740
	rel. err.	0.0227	0.0246	0.0230	0.0197	0.0160	0.0130	0.0111	0.0103	0.0096

Table 1: Exact implied vols σ , second order approximation $\bar{\sigma}_2$ and relative error $|(\bar{\sigma}_2 - \sigma)/\sigma|$. Parameters are the same as those in Figure 1.

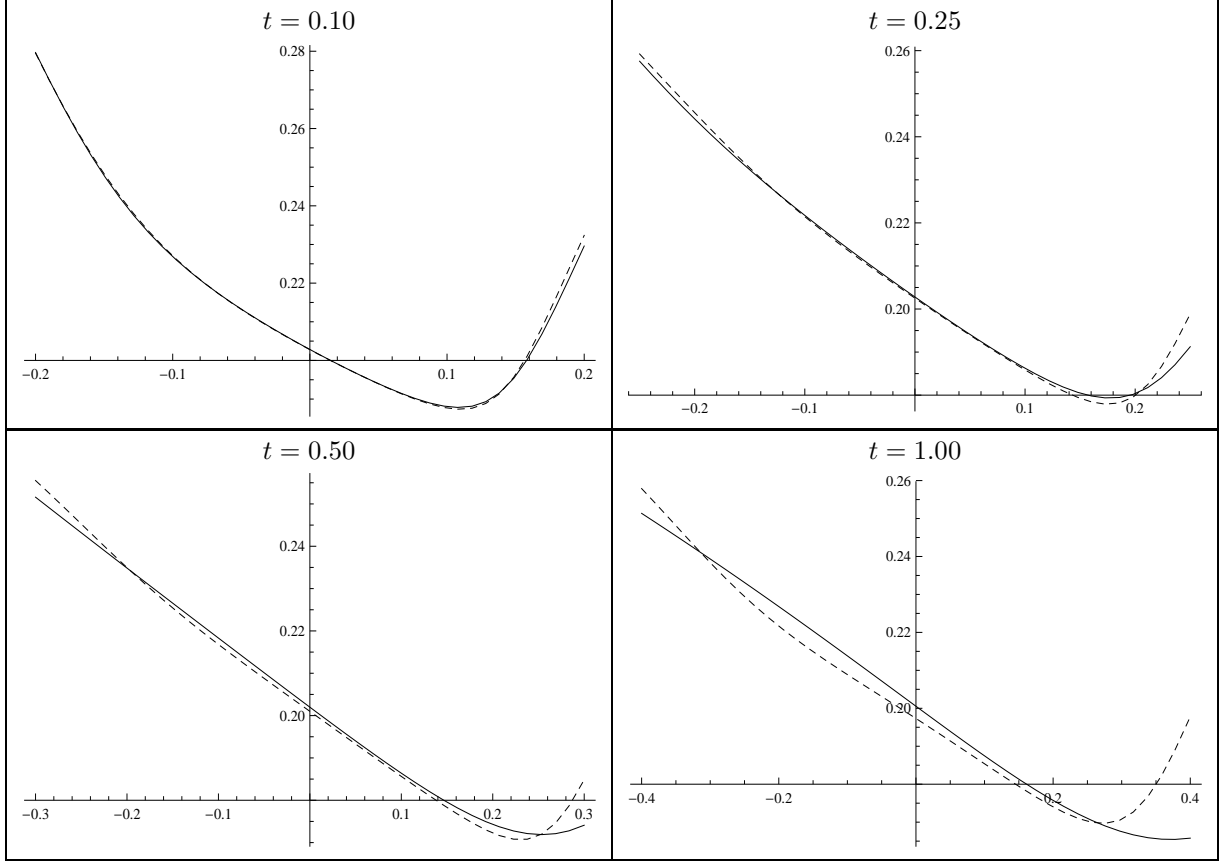
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$$\nu(d\zeta) = \frac{\lambda}{\sqrt{2\pi s^2}} \exp\left(\frac{-(\zeta - m)^2}{2s^2}\right).$$

Figure 1: For the model considered in Section 5, we plot the implied volatility σ corresponding to the exact option price u (solid black) as well as the implied volatility $\bar{\sigma}_2$ corresponding to our second order option price approximation \bar{u}_2 (dashed black). The units of the horizontal axis are log strike $k := \log K$. Approximate prices are computed using the Taylor series expansion of $\mathcal{A}(t)$ as described in Example 3.2. We assume the Lévy measure ν is as parametrized above. The following parameters are used in all four plots: $\kappa = 1.15$, $\theta = 0.04$, $\delta = 0.2$, $\rho = -0.7$, $z = \theta$, $x = 0$, $m = -0.1$, $s = 0.2$, $\lambda = 2.0$.

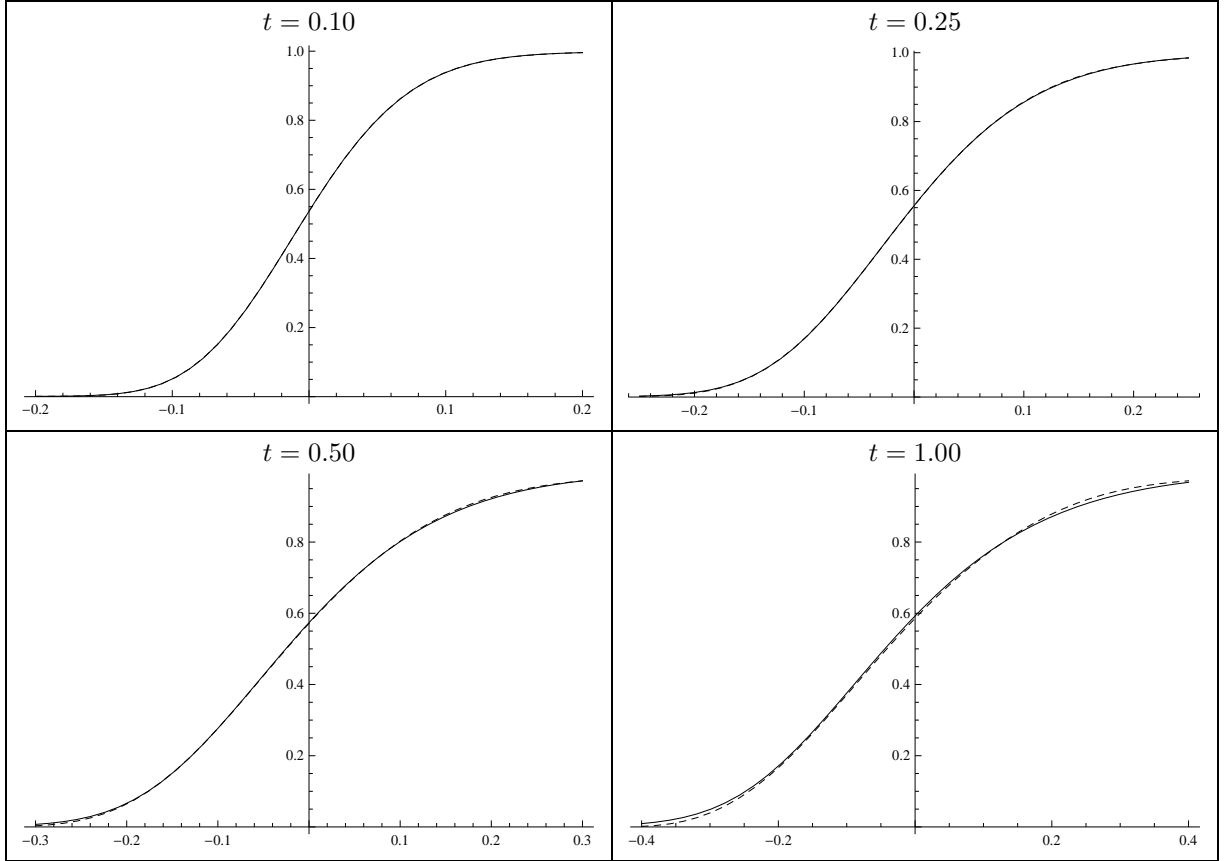


Figure 2: For the model considered in Section 5, we plot the Delta Δ corresponding to the exact option price u (solid black) as well as the Delta $\bar{\Delta}_2$ corresponding to our second order option price approximation \bar{u}_2 (dashed black). The units of the horizontal axis are x . Approximate prices are computed using the Taylor series expansion of $\mathcal{A}(t)$ as described in Example 3.2. We assume the Lévy measure ν is as given in Figure 1. The following parameters are used in all four plots: $\kappa = 1.15$, $\theta = 0.04$, $\delta = 0.2$, $\rho = -0.7$, $z = \theta$, $k = 0$, $m = -0.1$, $s = 0.2$, $\lambda = 2.0$.

	x	-0.2	-0.15	-0.1	-0.05	0.00	0.05	0.1	0.15	0.2
t=0.10	Δ	0.0008	0.00516	0.05084	0.2312	0.5370	0.8024	0.9385	0.9845	0.9959
	$\bar{\Delta}_2$	0.0009	0.00478	0.05081	0.2313	0.5368	0.8026	0.9387	0.9843	0.9958
	rel. err.	0.1309	0.07358	0.00048	0.0006	0.0003	0.0002	0.0002	0.0002	0.0000
t=0.25	Δ	0.01311	0.05708	0.1690	0.3503	0.5559	0.7329	0.8563	0.9293	0.9672
	$\bar{\Delta}_2$	0.0114	0.05674	0.1696	0.3502	0.5552	0.7330	0.8576	0.9306	0.9673
	rel. err.	0.1305	0.00585	0.0035	0.0004	0.0012	0.0000	0.0014	0.0014	0.0000
t=0.50	Δ	0.06608	0.1506	0.2767	0.4260	0.5739	0.7018	0.8014	0.8731	0.9215
	$\bar{\Delta}_2$	0.06425	0.1508	0.2766	0.4246	0.5719	0.7007	0.8027	0.8766	0.9256
	rel. err.	0.02773	0.0014	0.0003	0.0032	0.0034	0.0015	0.0015	0.0040	0.0044
t=1.00	Δ	0.1708	0.2667	0.3760	0.4878	0.5927	0.6849	0.7618	0.8234	0.8713
	$\bar{\Delta}_2$	0.1662	0.2627	0.3710	0.4814	0.5857	0.6791	0.7595	0.8262	0.8789
	rel. err.	0.0268	0.01496	0.0131	0.0130	0.0117	0.0084	0.0030	0.0033	0.0088

Table 2: Exact Delta Δ , second order approximation $\bar{\Delta}_2$ and relative error $|(\bar{\Delta}_2 - \Delta)/\Delta|$. Parameters are the same as those in Figure 2.

	x	-0.2	-0.15	-0.1	-0.05	0.00	0.05	0.1	0.15	0.2
t=0.10	Γ	0.01828	0.2978	2.159	5.539	6.288	3.831	1.446	0.3779	0.0780
	$\bar{\Gamma}_2$	0.01197	0.2897	2.1760	5.5300	6.288	3.841	1.437	0.3748	0.0821
	rel. err.	0.3452	0.0273	0.0077	0.0015	0.0001	0.0025	0.0061	0.0082	0.0518
t=0.25	Γ	0.5185	1.705	3.337	4.275	3.967	2.884	1.738	0.906	0.4229
	$\bar{\Gamma}_2$	0.5267	1.747	3.334	4.255	3.969	2.907	1.754	0.8925	0.4016
	rel. err.	0.0157	0.024	0.0009	0.0046	0.0003	0.0079	0.0094	0.0149	0.0503
t=0.50	Γ	1.514	2.488	3.135	3.206	2.802	2.174	1.54	1.017	0.635
	$\bar{\Gamma}_2$	1.585	2.508	3.109	3.182	2.804	2.208	1.588	1.045	0.6244
	rel. err.	0.0468	0.0079	0.0081	0.0076	0.0007	0.015	0.0309	0.0279	0.0167
t=1.00	Γ	2.095	2.425	2.483	2.306	1.985	1.612	1.251	0.9364	0.6814
	$\bar{\Gamma}_2$	2.134	2.418	2.452	2.280	1.988	1.656	1.331	1.028	0.7511
	rel. err.	0.0183	0.0032	0.0124	0.0110	0.0015	0.0276	0.0644	0.097	0.1023

Table 3: Exact Gamma Γ , second order approximation $\bar{\Gamma}_2$ and relative error $|(\bar{\Gamma}_2 - \Gamma)/\Gamma|$. Parameters are the same as those in Figure 3.

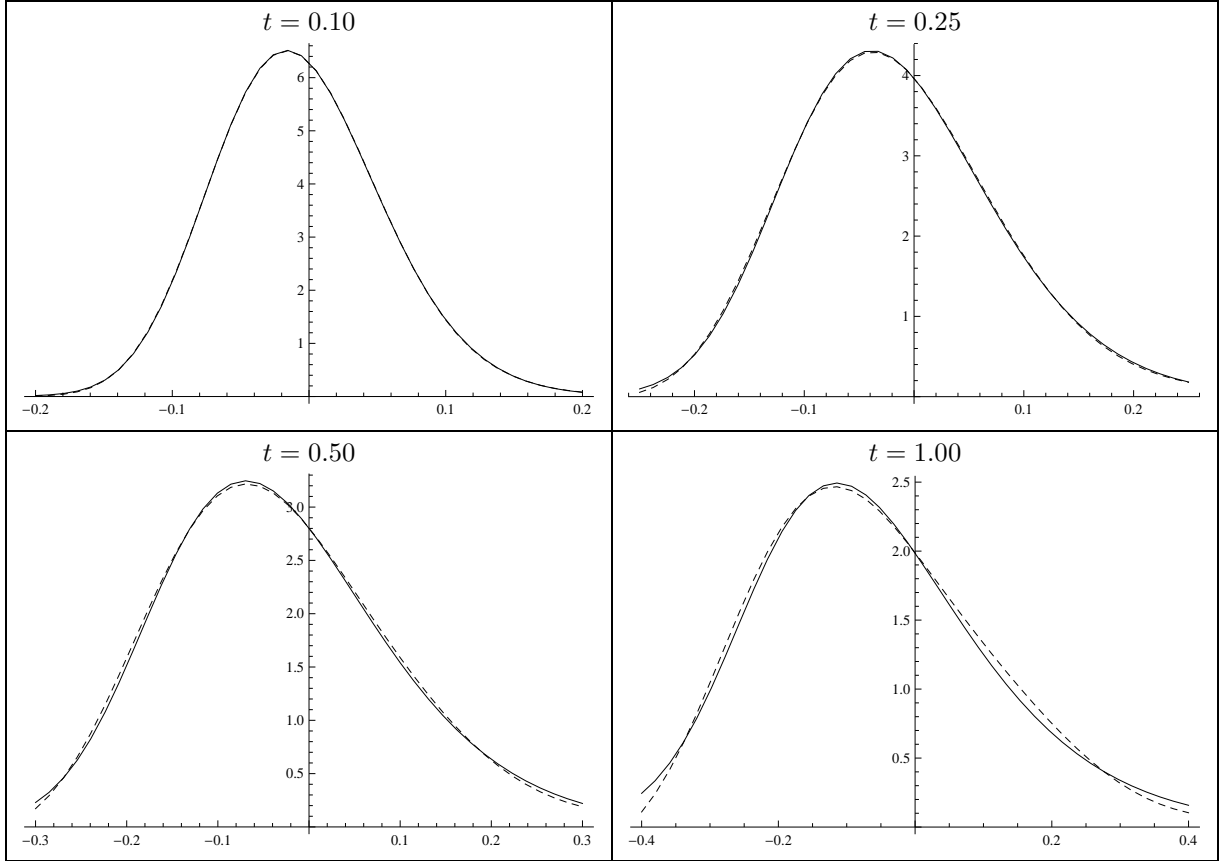


Figure 3: For the model considered in Section 5, we plot the Gamma Γ corresponding to the exact option price u (solid black) as well as the Gamma $\bar{\Gamma}_2$ corresponding to our second order option price approximation \bar{u}_2 (dashed black). The units of the horizontal axis are x . Approximate prices are computed using the Taylor series expansion of $\mathcal{A}(t)$ as described in Example 3.2. We assume the Lévy measure ν is as given in Figure 1. The following parameters are used in all four plots: $\kappa = 1.15$, $\theta = 0.04$, $\delta = 0.2$, $\rho = -0.7$, $z = \theta$, $k = 0$, $m = -0.1$, $s = 0.2$, $\lambda = 2.0$.